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AN ISOMORPHISM BETWEEN THETA CHARACTERISTICS AND THE $(2p + 2)$-POINT.*

By Arthur B. Coble.†

The theta functions of two and three variables have been studied in connection with respectively the set of 6 points on a conic and the set of 8 base points of a net of quadrics. The main object of this article is to show that there is a grouping of the theta functions of p variables which is isomorphic with the grouping of the simplest system of irrational invariants of a set of $2p + 2$ points in a projective space S_p. This isomorphism is set forth in § 2. Theta relations suggested by it are developed in § 4 for the case $p = 3$. In § 1 the tactical relations of the period and theta characteristics which appear most clearly in the light of finite geometry,‡ are developed in remarkably simple form with reference to a basis configuration. In § 3 a transition is made from the geometric to the arithmetic characteristic in order to determine the signs in a theta relation.

§ 1. The Characteristic Theory With Reference to a Basis Configuration.

In the finite space S_{2p-1} (mod. 2) let us take the supernumerary coördinates

$$(1) \qquad y_i \; (i = 1, \cdots, 2p + 1), \qquad \textstyle\sum_{i=1}^{i=2p+1} y_i = 0.$$

Then the bilinear relation between the points y and y',

$$(2) \qquad y_1 y_1' + y_2 y_2' + \cdots + y_{2p+1} y_{2p+1}' = 0,$$

is the equation of a null system C. The points y and y' are *syzygetic* or *azygetic* according as they do or do not satisfy (2). The points of S_{2p-1} are identified with the period characteristics (other than the zero characteristic) of the theta functions, the null system C with the bilinear relation on the periods, and the projectivities in S_{2p-1} with the integral linear transformations on the periods when reduced modulo 2.

The $2p + 1$ S_{2p-2}'s, $y_i = 0$, constitute an S_{2p-2}-*base* in the usual projective sense. As usual it determines a *point-base* ordered with regard to the S_{2p-2}-base. The point P_{0i} of the point-base associated with the S_{2p-2}, $y_i = 0$, has the coördinates, $y_i = 0$, $y_k = 1$ $(k \neq i)$. Evidently corresponding point and S_{2p-2} are point and null S_{2p-2} under C whence the two bases will be called a *self-dual base* of C and will be denoted by B_0. The further point, $P_{ik} = P_{ki} = P_{0i} + P_{0k}$ $(i \neq k; i, k = 1, \cdots, 2p+1)$, on the line $\overline{P_{0i}P_{0k}}$ has coördinates $y_i = y_k = 1$, $y_j = 0$ $(i \neq j \neq k)$ and is called a *residual* point of the base B_0. According to F. G., p. 271, the $(p+1)(2p+1)$ points P_{ik} $(i \neq k;\ i, k = 0, 1, \cdots, 2p+1)$ form the basis points and the residual points not only of the particular base B_0 but also of any one of $2p+2$ bases B_i $(i = 0, 1, \cdots, 2p+1)$ where the base B_i contains the $2p+1$ points P_{ik} for which $k \neq i$, $k = 0, 1, 2, \cdots, 2p+1$. These bases B_i also are self dual under C, the null S_{2p-2} of P_{ik} being $y_i + y_k = 0$. Thus the $2p+2$ bases B_i form a symmetrical *basis configuration* B and the point P_{ik} of B belongs to the bases B_i and B_k.

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‡ Coble, "An application of finite geometry etc.," Transactions of the American Mathematical Society, vol. XIV (1913), p. 241; referred to hereafter as F. G.

The remaining points of S_{2p-1} are determined in conformity with (1) by the following relations:

$$
\begin{aligned}
&P_{ij} + P_{jk} = P_{ik},\\
&\quad P_{ijkl} = P_{ij} + P_{kl} = P_{ik} + P_{jl} = \cdots,\\
(3)\qquad &\quad P_{ijklmn} = P_{ijkl} + P_{mn} = \cdots = P_{ij} + P_{kl} + P_{mn} = \cdots,\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&P_{012\cdots(2k-1)} = P_{(2k)(2k+1)\cdots(2p+1)},\\
&P_{012\cdots(2p+1)} = 0.
\end{aligned}
$$

That the $2^{2p} - 1$ points of S_{2p-1} are thus given, each in two ways, by the above notation is verified by

$$\binom{2p+2}{2} + \binom{2p+2}{4} + \cdots + \binom{2p+2}{2p} = \frac{1}{2}2^{2p+2} - 2 = 2(2^{2p} - 1).$$

It is clear from the form of (2) that

THEOREM 1. *Two points of S_{2p-1} are syzygetic or azygetic according as a symbol of one has an even or an odd number of subscripts in common with a symbol of the other.*

It is shown in F. G., p. 271, that the group of C contains a subgroup $G_{(2p+2)!}$ which permutes symmetrically the bases B_i. The collineations which correspond to the transpositions in the symmetric group are the involutions which are attached to the points of B as described in F. G., p. 246 (12). Then one sees at once from the above notation for the points that

THEOREM 2. *Under the $G_{(2p+2)!}$ of B the points of S_{2p-1} divide into $p/2$*

or $(p+1)/2$ *conjugate sets according as* p *is even or odd. The* k*-th set* $(k = 1, 2, \cdots, p/2$ *or* $(p+1)/2)$ *consists of the* $\binom{2p+2}{2k}$ *points whose symbol has* $2k$ *or* $2p+2-2k$ *subscripts; except in the case* $k = (p+1)/2$ *when the number is* $\frac{1}{2}\binom{2p+2}{p+1}$.

Let us now classify the 2^{2p} quadrics which belong to C (*i. e.*, whose polar systems coincide with C) according to their behavior under $G_{(2p+2)}$: According to F. G., p. 270 and p. 257, there is a unique quadric, $R_y = \sum y_i y_k$ $(i, k = 1, \cdots, 2p+1;\ i < k)$, which contains none of the residual points of B_0. The remaining quadrics are obtained from R_y by attaching any number of the squared terms $y_1^2, \cdots, y_{2p+1}^2$ to R_y. Since $\sum y_i^2 = 0$, any quadric can be expressed in two ways whence $\frac{1}{2}\, 2^{2p+2} = 2^{2p}$ quadrics are so obtained.

Consider the particular quadric, $R_y + y_1^2 + \cdots + y_p^2$. The point P_{ik} when either $i, k = 1, \cdots, p$ $(i < k)$ or $i, k = p+1, \cdots, 2p+1$ $(i < k)$ is not on R_y but is on $y_1^2 + \cdots + y_p^2$ and therefore is not on the quadric; while the point P_{ik} when $i = 1, \cdots, p$ and $k = p+1, \cdots, 2p+1$ is not on R_y nor on $y_1^2 + \cdots + y_p^2$ and therefore is on the quadric. The point P_{0k} is $\left\{\begin{matrix}\text{on}\\ \text{not on}\end{matrix}\right\}$ R_y if p is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$; the point P_{0k} is $\left\{\begin{matrix}\text{on}\\ \text{not on}\end{matrix}\right\}$ $y_1^2 + \cdots + y_p^2$ either if p is $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$ when $k < p+1$ or if p is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ when $k > p$; *i. e.*, P_{0k} is $\left\{\begin{matrix}\text{on}\\ \text{not on}\end{matrix}\right\}$ the quadric when $\left\{\begin{matrix}k > p\\ k < p+1\end{matrix}\right\}$ whether p be odd or even. The above quadric can be denoted by

$$Q_{0,1,\cdots,p} = Q_{p+1,\cdots,2p+1} = R_y + \sum_{i=1}^{i=p} y_i^2 = R_y + \sum_{i=p+1}^{i=2p+1} y_i^2$$

the notation indicating that it contains all the points P_{ik} of B for which i is drawn from $0, \cdots, p$ and k from $p+1, \cdots, 2p+1$. Again according to F. G., p. 257, R_y is an $\left\{\begin{matrix}E\\ O\end{matrix}\right\}$ quadric according as $p \equiv \left\{\begin{matrix}0, 3\\ 1, 2\end{matrix}\right\}$ mod. 2. The null point of $y_1 + \cdots + y_p = 0$ is on R_y if $\left\{\begin{matrix}\binom{p}{2}\\ \binom{p+1}{2}\end{matrix}\right\} \equiv 0$ mod. 2 when $p \equiv \left\{\begin{matrix}0\\ 1\end{matrix}\right\}$ mod. 2. Therefore in all cases $Q_{012\cdots p}$ is an E quadric. From its behavior with regard to the points of B we see that

THEOREM 3. *There exists for the basis configuration* B *a set of* $\frac{1}{2}\binom{2p+2}{p+1} = \binom{2p+1}{p}$ *median*[*] E *quadrics of the type* $Q_{0,1,\cdots,p} =$

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[*] The term "median" refers to the fact that the number of squares is as close to $\frac{1}{2}(2p+2)$ as possible.

$Q_{p+1,\,p+2,\,\cdots\,2p+1} = R_y + y_1^2 + \cdots + y_p^2$ *where* R_y *is the quadric on none of the residual points of* B_0. *Each is determined by a separation of the* $2p + 2$ *bases into two sets of* $p + 1$ *each and contains the* $(p + 1)^2$ *points* P_{ik} *of* B *such that* B_i *and* B_k *are drawn from different sets. This set of quadrics is conjugate under the* $G_{(2p+2)!}$ *of* B.

Of the two ways of writing a quadric, the one contains an even, the other an odd, number of squares. Let us for convenience write it so that the number of squares is even or odd with $p + 1$. Then the quadric of Theorem 3 is $Q_{0,1,\ldots,p} = Q_{p+1,\ldots,2p+1} = R_y + y_{p+1}^2 + \cdots + y_{2p+1}^2$. Furthermore every quadric can be expressed as $Q' = Q_{0,1,\ldots,p} + \Sigma_{i=1}^{i=2k} z_i^2$ where the z's are any $2k$ of the y's and where $0 \lesseqgtr k < p + 1$. Suppose to fix ideas that l of the z's are $y_{p-l+1}, \cdots, y_p$ and $2k - l$ are $y_{p+1}, \cdots, y_{p+2k-l}$. Then $\Sigma_{i=1}^{i=2k} z_i^2$ is the null S_{2p-2} of the point $P_{p-l+1,\ldots,p+2k-l}$. This point is $\left\{ \begin{matrix} \text{on} \\ \text{not on} \end{matrix} \right\}$ $Q_{0,1,\ldots,p}$ when $\binom{2k}{2} + l$ or when $k - l$ is $\left\{ \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right\}$. Since we know when the point P_{ij} lies on $Q_{01\ldots p}$, and when it lies on Σz_i^2 it is easy to verify that the point lies on Q' if one of its subscripts is drawn from the set $0, 1, \cdots, p - l, p + 1, \cdots, p + 2k - l$ and the other from the set $p - l + 1, \cdots, p, p + 2k - l + 1, \cdots, 2p + 1$.

THEOREM 4. *Every quadric belonging to* C *is uniquely determined by a separation of the bases of* B *into two sets of* $p + 1 - 2k$ *and* $p + 1 + 2k$ *each. It contains those and only those points* P_{ij} *of* B *which belong to bases* B_i *and* B_j *drawn from different sets. It is an* $\left\{ \begin{matrix} E \\ O \end{matrix} \right\}$ *quadric when* k *is* $\left\{ \begin{matrix} \textit{even} \\ \textit{odd} \end{matrix} \right\}$. *The particular quadric associated with the separation* $B_0, \cdots, B_{p-2k}; B_{p+1-2k}, \cdots, B_{2p+1}$ *is* $Q_{0,\ldots,p-2k} = Q_{p+1-2k,\ldots,2p+1} = R_y + \Sigma_{i=p+1-2k}^{i=2p+1} y_i^2 = 0$. *If* $p + 1$ *is even we have for* $k = (p + 1)/2$ *an isolated quadric* $Q = Q_{0,1,\ldots,2p+1}$ *which contains none of the points of* B. *Under the* $G_{(2p+2)!}$ *of* B *the quadrics divide into* $\left\{ \begin{matrix} (p + 1)/2 \\ p/2 \end{matrix} \right\}$ *conjugate sets if* p *is* $\left\{ \begin{matrix} \textit{odd} \\ \textit{even} \end{matrix} \right\}$.

E. g., when $p = 2$ we have 10 E quadrics $Q_{ijk} = Q_{lmn}$ and 6 O quadrics $Q_i = Q_{jklmn}$. When $p = 3$ we have 35 E quadrics $Q_{ijkl} = Q_{mnop}$, 28 O quadrics $Q_{ij} = Q_{klmnop}$, and 1 E quadric $Q = Q_{01234567}$.

The question as to the incidence of a point P whose symbol has $2s$ subscripts with the quadric $Q_{0,\ldots,p-2k}$ is easily answered. Let the set of $2s$ subscripts of P be that set which does not contain 0 and of these let r be found in the set $0, \cdots, p - 2k$. Then the result of substituting in the equation of the quadric as given in Theorem 4 is $\binom{2s}{2} + 2s - r$ which is even or odd with $s + r$. But $s + r$ is even or odd with $s + (2s - r)$, or with $(p + 1 - s) + (p + 1 + 2k - r)$, or with $(p + 1 - s) + (p + 1 - 2s - 2k + r)$ so that

THEOREM 5. *A point lies on a quadric if half the number of subscripts in a symbol for the point together with the number of subscripts common to this symbol and a symbol for the quadric is even.*

E. g., when $p = 3$ the E quadric $Q_{ijkl} = Q_{mnop}$ contains the 16 points $P_{im} = P_{jklnop}$, the 18 points $P_{ijmn} = P_{klop}$ and the point $P_{ijkl} = P_{mnop}$. The O quadric $Q_{ij} = Q_{klmnop}$ contains the 12 points $P_{ik} = P_{jlmnop}$ and the 15 points $P_{ijkl} = P_{mnop}$. The E quadric $Q = Q_{ijklmnop}$ contains the 35 points $P_{ijkl} = P_{mnop}$.

The result of adding to the quadric above the square of the null S_{2p-2} of the point P is evidently a quadric for which the $2s - r$ common squares reduce to zero mod. 2 so that the new quadric contains $r + (p + 1 + 2k) - (2s - r)$ squares. Hence

THEOREM 6. *The symbol for the quadric obtained by adding to a given quadric the square of the null S_{2p-2} of a given point has for its subscripts the aggregate of the subscripts in a symbol of the given quadric and a symbol of the given point less the subscripts common to the two given symbols.*

The group of the null system is generated by a conjugate set of involutions $I_{r_1, r_2, \dots}$ each associated with a point $P_{r_1, r_2, \dots}$. The involution $I_{r_1, r_2, \dots}$ transforms the point $P_{s_1, s_2, \dots}$ into itself if $P_{r_1, r_2, \dots}$ and $P_{s_1, s_2, \dots}$ are syzygetic; otherwise into the point $P_{r_1, r_2, \dots} + P_{s_1, s_2, \dots} = P_{r_1, r_2, \dots; s_1, s_2, \dots}$, where like subscripts cancel each other when adjoined. It is often convenient to effect these generating involutions upon the quadrics attached to the null system and this is accomplished as follows:

THEOREM 7. *The involution determined by a given point P with $2k$ subscripts transforms a given quadric whose symbol has σ subscripts in common with P, into itself when $k + \sigma$ is odd; but when $k + \sigma$ is even into a quadric whose symbol has for subscripts the sum of those of the given point and quadric.*

To prove this let $I_{r_1, r_2, \cdots, r_\sigma; s_1, s_2, \cdots, s_{2k-\sigma}}$ be the given involution and let the given quadric be

$$Q_{r_1, \cdots, r_\sigma; t_1, \cdots, t_{p+1-2l-\sigma}} = Q_{s_1, \cdots, s_{2k-\sigma}; u_1, \cdots, u_{p+1+\sigma+2l-2k}}.$$

Then Q contains the four sets of basis points

$$(a)\ P_{r,s},\ (b)\ P_{t,u},\ (c)\ P_{t,s},\ (d)\ P_{r,u},$$

which are transformed by I into respectively

$$(a)\ P_{r,s},\ (b)\ P_{t,u},\ (c)\ P_{t,\bar{r},\bar{s}'},\ (d)\ P_{u,\bar{r}',\bar{s}}.$$

Here a subscript r stands for any one of the r's, $\bar{r}$ for all the r's and $\bar{r}'$ for all but one of the r's. Applying the rule of Theorem 5 we see that if $k + \sigma$ is odd the given quadric contains all four transformed sets, and that if $k + \sigma$ is even the quadric $Q_{s_1, \cdots, s_{2k-\sigma};\, t_1, \cdots, t_{p+1-2l+\sigma}}$ contains all four transformed sets. Since there is one and only one quadric on the original four sets the transformed quadric on the transformed set is uniquely determined.

It seems that the above notation for the period and theta characteristics can hardly be improved. For not only is the notation (3) for the period characteristics and the notation of Theorem 4 for the theta characteristics each remarkably simple; but also the various fundamental conditions such as, (3) for the addition of half-periods, Theorem 1 for syzygetic or azygetic period characteristics, Theorem 4 for even or odd theta characteristics, Theorem 5 for the vanishing or non-vanishing of a given theta function for a given half-period, Theorem 6 for the theta function obtained by adding to the argument of a given theta function a given half-period, and Theorem 7 for the transformation of the theta functions, all are extremely easy to apply.

§ 2. An Isomorphism between the Median E Quadrics of B and the $(2p + 2)$-point in S_p.

The median E quadrics of Theorem 3 admit of a grouping into sets of $p + 2$ which appears as follows. Consider a $(p + 2, p)$ separation of the bases of B into a set $B_0, \cdots, B_{p+1}$ and a set $B_{p+2}, \cdots, B_{2p+1}$. If a base of the first set be added to the second a $(p + 1, p + 1)$ separation is obtained which determines a median E quadric. If a base of the second set be added to the first set, the $(p + 3, p)$ separation determines as in Theorem 4 a median O quadric. The $p + 2$ median E quadrics and the p median O quadrics so derived constitute a fundamental system ($F.\ S.$) of $2p + 2$ quadrics. In fact these quadrics can be gotten by beginning with $R_y + \Sigma_{i=1}^{i=p+1} y_i^2$ and adding in turn $y_1^2, \cdots, y_{2p+1}^2$. By referring to the table (F. G., p. 272) we see that in any $F.\ S.$ the number s of O quadrics is congruent to p mod. 4. When $s = p$ the values of k in the table are $p/2$; $(p + 2)/2$ or $(p - 1)/2$; $(p + 1)/2$ or $(p + 2)/2$; $p/2$ or $(p + 1)/2$; $(p - 1)/2$ according as $p \equiv 0, 1, 2, 3$ mod. 4. Hence both the E and the O quadrics in the $F.\ S.$ are median quadrics of the attached B.

THEOREM 8. *The* $\frac{1}{2}\binom{2p+2}{p+1}$ *median E quadrics of a basis configuration B can be grouped in* $\binom{2p+2}{p}$ *ways into sets of $p + 2$ each such that each set can be supplemented by p median O quadrics to form a $F.\ S.$ of $2p + 2$ quadrics attached to B. Every $F.\ S.$ containing precisely p O quadrics can be obtained in this way.*

Consider now $2p+2$ points in a projective space S_p numbered 0, 1, $\cdots$, $2p+1$. One can form from them $\frac{1}{2}\binom{2p+2}{p+1}$ determinant products of the type $(0, 1, \cdots, p)(p+1, p+2, \cdots, 2p+1)$. This product corresponds in notation to the median E quadric $Q_{0,1,\cdots,p} = Q_{p+1,\cdots,2p+1}$. If we separate the points into a set of $p+2$ and a set of p, a determinant product can be formed by taking a point of the first set and adding it to the second set. Moreover the $p+2$ products thus formed satisfy a linear relation. Hence

THEOREM 9. *The* $\frac{1}{2}\binom{2p+2}{p+1}$ *determinant products that can be formed from* $2p+2$ *points in* S_p *are grouped* $p+2$ *at a time in* $\binom{2p+2}{p}$ *determinant identities precisely as the median* E *quadrics of Theorem* 8 *are grouped in* F. S.'s *containing* p *median* O *quadrics.*

For the case $p = 2$ it appears* that this formal isomorphism is vitalized by the existence of theta relations (*i. e.*, linear relations connecting the squares of the functions) which parallel the determinant identities and which lead to the determination of a set of 6 points in terms of theta modular functions. A similar set of relations is derived in § 4 for $p = 3$.

§ 3. The Determination of the Signs in Theta Relations.

In the usual notation for the theta function a proper half-period is given by a scheme of $2p$ numbers $\epsilon = \begin{pmatrix} \epsilon_1, & \cdots, & \epsilon_p \\ \epsilon_{p+1}, & \cdots, & \epsilon_{2p} \end{pmatrix}$ which are 0 or 1 but not all zero. This we have identified with the point x in the finite space S_{2p-1} such that $x_i = \epsilon_i$. The notation for the characteristic of an odd or even theta function of the first order is similarly $\eta = \begin{pmatrix} \eta_1, & \cdots, & \eta_p \\ \eta_{p+1}, & \cdots, & \eta_{2p} \end{pmatrix}$, $\eta_i = 0, 1$. This has been identified with the quadric in S_{2p-1}

$$(1) \quad x_1x_{p+1} + \cdots + x_px_{2p} + \eta_1{x^2}_{p+1} + \eta_{p+1}x_1^2 + \cdots + \eta_px_{2p}^2 + \eta_{2p}x_p^2 = 0.$$

The null system C, or invariant relation between two half-periods x and x', is then

$$x_1x'_{p+1} + x_{p+1}x_1' + \cdots + x_px_{2p}' + x_{2p}x_p' = 0.$$

In determining the coefficients of a theta relation one often adds a half-period to the argument. The change in a term is then given by the formula,†

$$\vartheta[\eta](u+\epsilon) = f(\epsilon) \cdot e^{-(\epsilon_1\eta_{p+1}+\cdots+\epsilon_p\eta_{2p})/4} \cdot \vartheta[\eta+\epsilon](u).$$

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* Coble, "Point Sets and Allied Cremona Groups," Transactions of the American Mathematical Society, vol. XVI (1915), p. 155, § 9; referred to hereafter as P. S.

† Krazer, Lehrbuch der Thetafunktionen, p. 240 (VII).

Here $f(\epsilon)$, an exponential factor, depends only on the half-period ϵ and disappears from the relation; $\vartheta[\eta + \epsilon]$ corresponds in F. G. to the quadric obtained by adding to the quadric corresponding to $\vartheta[\eta]$ the square of the null S_{2p-2} of the point corresponding to ϵ, while the factor $e^{-(\Sigma\epsilon_i\eta_{p+i})/4}$ gives rise in any relation involving the squares of the thetas to ± 1 according as the congruence,

$$\epsilon_1\eta_{p+1} + \cdots + \epsilon_p\eta_{2p} \equiv 0 \text{ mod. } 2, \tag{2}$$

is or is not satisfied. But in general we are interested only in the ratio of two such signs say those determined by the terms $[\eta]$ and $[\eta']$. These are like or unlike according as

$$K = \epsilon_1\epsilon'_{p+1} + \cdots + \epsilon_p\epsilon_{2p}' \equiv 0 \text{ mod. } 2 \tag{3}$$

is or is not satisfied when $(\epsilon') = [\eta] + [\eta']$. This relation (3) on the points ϵ, ϵ' of our F. G. is a correlation K which we wish to study in connection with the given null system C.

Let us first notice that the two spaces, S_{p-1}:

$$G_1, \text{ defined by } x_{p+1} = x_{p+2} = \cdots = x_{2p} = 0, \text{ and}$$

$$G_2, \text{ defined by } x_1 = x_2 = \cdots = x_p = 0;$$

are such that the null S_{2p-2} of any point in the one contains it completely; i. e., each is a null space of C or a *Göpel space* as defined in F. G., p. 248. Points of G_2 taken as points ϵ, and points of G_1 taken as points ϵ', are singular points of K. But a point of G_1 taken as ϵ has a corresponding S_{2p-2}, the same under K as under C, which cuts G_2 in an S_{p-2}; and a point of G_2 taken as ϵ' has a corresponding S_{2p-2} under K or under C which cuts G_1 in an S_{p-2}. Hence

THEOREM 10. *The null system C determines a proper correlation between any two skew Göpel spaces. Conversely given a correlation between any two skew S_{p-1}'s in S_{2p-1}, this is determined by a unique null system C for which the S_{p-1}'s are Göpel spaces. The two Göpel spaces are contained in one and only one E quadric belonging to C.*

For if G_1, G_2, and K are given, a point x of G_1 determines under K an S_{p-2} in G_2 which is joined to G_1 itself by an S_{2p-2}, the null S_{2p-2} of x under C. The null S_{2p-2} of x' in G_2 under C is determined similarly. Any point not on G_1 or G_2 can be expressed in a single way as $x + x'$ whence its null S_{2p-2} under C is known and C is determined. Moreover it is clear from (1) that only the one quadric $[\eta] = 0$ contains G_1 and G_2.

If then there be isolated in the finite geometry a pair of skew Göpel spaces, the original E function $\vartheta(u)$ is isolated, the correlation K is determined, and thereby the signs in the theta relations are determined. Some choices of G_1 and G_2 particularly convenient for the notation of § 1 will now be given.

In the space G_1 with coördinates $x_1, \cdots, x_p$ let the reference points in order and the unit point be denoted by $g_0, g_1, \cdots, g_p$; in G_2 with coordinates $x_{p+1}, \cdots, x_{2p}$ let the similar points be denoted by $\gamma_0, \gamma_1, \cdots, \gamma_p$. Then $\Sigma g = 0$ and $\Sigma\gamma = 0$. In the notation of § 1 consider the two sets of $p+1$ points each:

$$\begin{array}{ll} & P_{01}, P_{23}, P_{45}, \cdots, P_{2p,\,2p+1}; \\ (4) & \\ & P_{12}, P_{34}, P_{56}, \cdots, P_{2p+1,\,0}. \end{array}$$

The sum of the points in each set is identically zero so that each determines an S_{p-1}. Any two points of a set are syzygetic so that the S_{p-1}'s are Göpel spaces. No point of the one space can be a point of the other for if the even subscripts of two such points coincide the odd ones cannot. Hence the sets (4) can be identified with the sets g of G_1 and γ of G_2. The quadric corresponding to $\vartheta(u)$ is the median E quadric $Q_{0,\,2,\,4,\,\cdots,\,(2p)} = Q_{1,\,3,\,5,\,\cdots,\,(2p+1)}$.

The expression for any given point $P_{r_1,\,r_2,\,\cdots}$ in terms of the above points of G_1 and G_2 is quickly obtained by arranging $r_1, \cdots, r_{2k}$ in ascending order and filling the intervals between them with the intervening numbers each taken twice. E. g., $P_{1347} = P_{1223455667} = P_{12} + P_{23} + P_{45} + P_{56} + P_{67} = (P_{23} + P_{45} + P_{67}) + (P_{12} + P_{56}) = \begin{pmatrix} 011100\cdots \\ 101000\cdots \end{pmatrix}$. If two points P, P' are so expressed in the form $P = g + \gamma$, $P' = g' + \gamma'$ then P and P' satisfy K if g and γ' satisfy C.

When p is odd and the $2p+2$ subscripts are divided into two sets ordered with regard to each other, say $\begin{pmatrix} 0, & 1, & \cdots, & p \\ p+1, & p+2, & \cdots, & 2p+1 \end{pmatrix}$, then the $p+1$ points formed by taking p subscripts from the one set and the $(p+1)$th complementary subscript from the other are such that their sum is identically zero and that any two are syzygetic whence they determine a G_1. The general point of G_1 is formed from j subscripts of the one line and $\left\{\begin{matrix} p+1-j \text{ complementary} \\ j \text{ corresponding} \end{matrix}\right\}$ subscripts of the other when j is $\left\{\begin{matrix} \text{odd} \\ \text{even} \end{matrix}\right\}$. By shifting the subscripts of the lower row forward one space a Göpel space G_2 is similarly determined which as before is skew to G_1. In this case the quadric containing G_1 and G_2 is the median E quadric $Q_{0,\,1,\,\cdots,\,p} = Q_{p+1,\,\cdots,\,2p+1}$ when $p \equiv 1 \pmod{4}$; but if $p \equiv 3 \pmod{4}$ it is $Q = Q_{0,\,1,\,\cdots,\,2p+1}$ (see (8)).

For the case $p = 2$ a pair of Göpel spaces (4) was used in P. S., p. 191.

For the case $p = 3$ a pair of Göpel spaces of the other sort is used in the next paragraph.

§ 4. Some Theta and Theta-modular Relations for $p = 3$.

We denote by $\pi_{r_1, r_2, \ldots}$ the half-period which corresponds to $P_{r_1, r_2, \ldots}$, and by $\vartheta_{r_1, r_2, \ldots}(u)$ the theta function which corresponds to $Q_{r_1, r_2, \ldots}$. Also let $\vartheta_{r_1, r_2, \ldots} = [\vartheta_{r_1, r_2, \ldots}(u)]_{u=0}$. The 5 E quadrics Q_{jklm}, Q_{iklm}, Q_{ijlm}, Q_{ijkm}, Q_{ijkl} together with the three O quadrics Q_{op}, Q_{pn}, Q_{no} lie in a $F. S.$ (§ 2). Since there is a linear relation connecting any 9 theta squares, the 8 squares corresponding to the above quadrics together with $\vartheta^2(u)$ must be linearly related. The point P_{op} does not lie on any of the E quadrics nor on Q_{op} but does lie on Q_{pn} and Q_{no}. If then we substitute π_{op} for u in the assumed relation all the terms vanish except that containing $\vartheta_{op}(u)$ whence that term cannot occur. Similarly none of the odd thetas can occur and the form of the identity is

$$a\vartheta^2(u) - a_{jklm}\vartheta^2_{jklm}(u) - a_{iklm}\vartheta^2_{iklm}(u) - a_{ijlm}\vartheta^2_{ijlm}(u)$$
$$- a_{ijkm}\vartheta^2_{ijkm}(u) - a_{ijkl}\vartheta^2_{ijkl}(u) = 0.$$

The point P_{jklm} is on Q and Q_{jklm} but is not on the other E quadrics whence for $u = \pi_{jklm}$ we get

$$a\vartheta^2(\pi_{jklm}) = a_{jklm}\vartheta^2_{jklm}(\pi_{jklm}).$$

From Theorem 6 we can set to within sign $a = \vartheta^2$, $a_{jklm} = \vartheta^2_{jklm}$. Reverting to (2) and (3) of § 3 if $\epsilon = \pi_{jklm} = \begin{pmatrix} \epsilon_1\epsilon_2\epsilon_3 \\ \epsilon_4\epsilon_5\epsilon_6 \end{pmatrix}$, then, since η = characteristic of $\vartheta(u) = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}$, we see that $\eta_{jklm} = \begin{pmatrix} \epsilon_1\epsilon_2\epsilon_3 \\ \epsilon_4\epsilon_5\epsilon_6 \end{pmatrix}$. Thus $\eta + \eta_{jklm} = \epsilon' = \epsilon$ and the condition K in (3) § 3 is satisfied since P_{jklm} is on Q. Hence the signs of a and a_{jklm} are like and we see that

THEOREM 11. *After isolating the even theta function $\vartheta(u)$, the squares of the remaining* 35 *even thetas are linearly related with $\vartheta^2(u)$ by means of* 56 *identities of the type*

$$\vartheta^2\vartheta^2(u) - \vartheta^2_{jklm}\vartheta^2_{jklm}(u) - \vartheta^2_{iklm}\vartheta^2_{iklm}(u) - \vartheta^2_{ijlm}\vartheta^2_{ijlm}(u)$$
$$- \vartheta^2_{ijkm}\vartheta^2_{ijkm}(u) - \vartheta^2_{ijkl}\vartheta^2_{ijkl}(u) = 0.$$

Let us denote by $\overline{\overline{ijkl}\,\overline{mnop}}$ the determinant product formed from the coördinates of eight points $p_0, \cdots, p_7$ in S_3 when in each determinant the points are arranged in the natural order; and by $\epsilon_{ijklmnop}$ the sign of the permutation $ijklmnop$ after such an arrangement has been made. Now a typical pair of terms in a determinant identity is $\overline{ijkl}\,\overline{mnop} - \overline{ijkm}\,\overline{lnop}$.

Let ϵ_1 and $\epsilon_2 = -\epsilon_1$ be the signs of the permutations as written, and let a_1 and a_2 be respectively the number of inversions which are necessary to naturalize the order in the determinants of a product. Then $\epsilon_{ijklmnop} = (-1)^{a_1}\epsilon_1$ and $\epsilon_{ijklmnop} = (-1)^{a_2}\epsilon_2$. Hence the pair of terms

$$\epsilon_{ijklmnop}\,\overline{\overline{ijkl}}\;\overline{\overline{mnop}} + \epsilon_{ijkmlnop}\,\overline{\overline{ijkm}}\;\overline{\overline{lnop}}$$

is

$$(-1)^{a_1}\epsilon_1\,\overline{\overline{ijkl}}\;\overline{\overline{mnop}} + (-1)^{a_2}\epsilon_2\,\overline{\overline{ijkm}}\;\overline{\overline{lnop}}$$

or

$$(-1)^{2a_1}\epsilon_1\,\overline{ijkl}\;\overline{mnop} - (-1)^{2a_2}\epsilon_1\,\overline{ijkm}\;\overline{lnop}$$

or $\epsilon_1[\overline{ijkl}\;\overline{mnop} - \overline{ijkm}\;\overline{lnop}]$. Hence

THEOREM 12. *The 35 combinations of theta squares,* $\frac{1}{5}\vartheta^2\vartheta^2(u) - \vartheta^2_{ijkl}\vartheta^2_{ijkl}(u)$; *the 35 theta modular functions,* $\frac{1}{5}\vartheta^4 - \vartheta^4_{ijkl}$; *and the 35 determinant products formed from 8 points in* S_3, $\epsilon_{ijklmnop}\,\overline{\overline{ijkl}}\;\overline{\overline{mnop}}$; *each satisfy the same set of 56 relations typified by*

$$[ijkl] + [ijkm] + [ijlm] + [iklm] + [jklm] = 0.$$

Since the 8 points in S_3 when self-associated, i. e., when they are the base points of a net of quadrics, depend upon 6 absolute constants, and the theta modular functions also depend upon 6 moduli it seems likely that the equations $\epsilon_{ijklmnop}\,\overline{\overline{ijkl}}\;\overline{\overline{mnop}} = \frac{1}{5}\vartheta^4 - \vartheta^4_{ijkl}$ would serve for a parametric expression for 8 self-associated points in terms of the theta modular functions. Before this conclusion could be drawn however it would be necessary to show that the relations of higher degree satisfied by the determinant products (of which fourteen are linearly independent), and the relations (P. S., p. 165) which imply self-association, are likewise satisfied by the corresponding modular functions.

The 56 relations of Theorem 11 are associated with the basis configuration B which isolates $\vartheta(u)$. There being 36 B's we should expect to find 36.56 relations each occurring 6 times. There ought then to be 5.56 relations whose subscripts take another form. If we apply to B the involution I_{ijkn} the quadrics and points considered above are transformed into similarly situated quadrics and points. We can therefore infer that there is a linear identity connecting the six transformed even thetas which we shall prove is as follows:

$$\begin{aligned}(1)\qquad &\vartheta^2_{ijkl}\vartheta^2_{ijkl}(u) + \vartheta^2_{ijkm}\vartheta^2_{ijkm}(u) + \vartheta^2_{ijkn}\vartheta^2_{ijkn}(u)\\ &\qquad = \vartheta^2_{ilmn}\vartheta^2_{ilmn}(u) + \vartheta^2_{jlmn}\vartheta^2_{jlmn}(u) + \vartheta^2_{klmn}\vartheta^2_{klmn}(u).\end{aligned}$$

We have only to verify the coefficients. If $u = \pi_{lm}$ all the terms except the first two vanish. Here $\eta = [ijkl]$, $\eta' = [ijkm]$, and $\epsilon' = \eta + \eta' =$

$[lm]$ while $\epsilon = [lm]$ also. Hence K in (3) § 3 is not satisfied since P_{lm} is not on Q. Then to within a common factor the terms become $\vartheta^2_{ijkl}\vartheta^2_{ijkm} - \vartheta^2_{ijkm}\vartheta^2_{ijkl}$ which checks the second coefficient. The third can be similarly checked. If $u = \pi_{jkmn}$ all terms vanish except the first and fourth. Then $\eta = [ijkl]$, $\eta' = [ilmn]$, $\epsilon' = \epsilon = [jkmn]$ and K is satisfied since P_{jkmn} is on Q. To within a factor the relation reduces to $\vartheta^2_{ijkl}\vartheta^2_{ilmn} = \vartheta^2_{ilmn}\vartheta^2_{ijkl}$ which checks the fourth coefficient. The fifth and sixth coefficients can be verified similarly. The relation (1) depends upon the separation of $ijk, lmn = lmn, ijk$ from 8 things whence there are 56.5 of this form.

THEOREM 13. *The* 56.6 *theta relations for* $p = 3$ *which contain the five even functions which lie in any F. S. with three odd functions are comprised in the* 56 *relations of Theorem* 11 *and the* 56.5 *relations of type* (1).

If we denote the relation of Theorem 11 when transposed to the right by $\{ijklm\} = 0$ and (1) when transposed to the left by $\{ijk, lmn\} = 0$, then

$$\{ijkmn\} + \{ijknl\} + \{ijklm\} - \{lmnjk\} - \{lmnki\} - \{lmnij\} = 2\{ijk, lmn\}.$$

Thus none of the relations (1) are independent of those of Theorem 11. By means of the latter relations alone the 36 even theta squares can be expressed in terms of 15.

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